Formulation of time-varying throughput flow for single-lane highways considering traffic mixed with human-driven and automated vehicles

Formulação do fluxo de tráfego variável em rodovias de faixa única, considerando tráfego misto de veículos autônomos e regulares

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Daone da Silva Santos
Bacharel em Engenharia Civil
Instituição: Instituto Federal de Alagoas – IFAL/Campus Piranhas
Endereço: Av. Sergipe 1477, Bairro de Xingó, Piranhas-AL, CEP 57.460-000
E-mail: daone.santos@ifal.edu.br

José Lucas de Oliveira Gregório
Graduando em Análise e Desenvolvimento de Sistemas
Instituição: Faculdade de Tecnologia do Estado de São Paulo
Endereço: Rua Frederico Grotte, 322, São Paulo - SP
E-mail: jose.gregorio@fatec.sp.gov.br

Moabio Elizandro Rodrigues Barreto Filho
Graduando em Licenciatura em Física
Instituição: Instituto Federal de Alagoas – IFAL/Campus Piranhas
Endereço: Av. Sergipe 1477, Bairro de Xingó, Piranhas-AL, CEP 57.460-000
E-mail: moabiorodrigues.2001@gmail.com

ABSTRACT
Connected and automated vehicle (AV) technologies are promising to improve traffic efficiency, stability, and safety. Mixed traffic of AVs and Human-driven vehicles (HDVs) consists of a topic of great interests among many nations worldwide, and it may be real in the near future. In this matter, this paper delivers a formulation for estimating time-varying throughput flows in single-lane highways based on a car-following model with automated and human-driven vehicles. The formulation considers the inter-vehicle spacing characteristics and it is derived based upon rectilinear motion equations in which accelerations are implemented within bounds. Formulations for spacing characteristics and throughput flows are derived in terms of HDVs accelerations, and the limit intervals of the HDVs’ accelerations variations are proposed, delineating two states of driver awareness: state of distraction and state of alertness. In addition, numerical implementations are performed, and the results obtained from the proposed formulations are compared with those obtained by Chen et al. (2017). The formulations developed in this study showed to be propitious to the understanding of mixed traffic since they permit adjustments to consider varying driving behaviors among HDVs and differences in vehicle characteristics.

Keywords: Mixed Traffic, Capacity, Automated Vehicles.
RESUMO
Tecnologias referentes aos veículos autônomos e conectados (AV) são promissoras para a melhoria da eficiência, estabilidade e segurança do tráfego. Tráfego misto com AVs e veículos regulares (HDVs) consiste em um tópico de grande interesse entre muitas nações e poderá tornar-se real em breve. Neste contexto, este trabalho propõe uma fórmula para estimar os fluxos de tráfego variável em rodovias de faixa de rolamento única, baseada em um modelo car-following disposto de tráfego misto de AVs e HDVs. A fórmula proposta neste trabalho considera características de espaçamentos entre veículos e baseia-se em equações de movimento retílineo nas quais as acelerações são implementadas dentro de limites. As fórmulas para as características de espaçamento e o fluxo de tráfego são desenvolvidas em termos das acelerações dos veículos regulares, e limites de variações para as acelerações dos veículos regulares são propostos considerando dois estados hipotéticos de percepção dos condutores: estado de distração e estado de alerta. Em adição, o modelo numérico é implementado e os resultados obtidos com as formulações propostas são comparados com resultados advindos de Chen et al. (2017). As formulações desenvolvidas neste trabalho se mostraram propícias para o entendimento de um cenário de tráfego misto já que elas permitem ajustes para incorporar diferentes comportamentos de condutores dos veículos regulares, bem como características de aceleração diferentes entre estes veículos.

Palavras chave: Tráfego Misto, Capacidade, Veículos Autônomos.

1 INTRODUCTION
The world is drastically changing and experiencing a genesis in the mobility sector due to some emerging connected and automated vehicle technologies. These technologies are expected to enable a completely new traffic pattern while improving traffic efficiency, stability, and safety, radically changing driver interactions, and enabling extraordinary opportunities. In the last decade the efforts in the context of connected and automated vehicles have expanded tremendously due to the tireless endeavors of the triad academy-industry-government, such that, mixed traffic of AVs and HDVs – named mixed traffic hereafter - is not a matter of “if” but rather “how soon” (Bagloee et al., 2017).

Accordingly, numerous studies and research programs suggest that by 2030, AVs will be on course, radically changing driver interactions and transforming the automobile industry (Hendrickson, 2014; McCarthy et al., 2015). In this context, this oncoming scenario will, inevitably, have to accommodate mixed traffic, which means that AVs will have to co-exist with HDVs (Sharma et al., 2017). It forces great research efforts to formulate and understand such complex mixed traffic flow condition.

Vehicle platooning consists of a notably opportune technology, very suitable in the context of AVs, which can improve roadway capacity and flow stability significantly.
(Milanés et al., 2014; Milanés and Shladover, 2014; Shladover et al., 2012). Some research studies have dedicated efforts concerning platoon size, reaction time, acceleration/deceleration, regarding AVs and their impacts on roadway capacity (Zhao and Sun, 2013; Talebpour et al., 2016; Talebpour et al., 2017). In respect to that, Chen et al. (2017) developed formulations of traffic operational capacity in mixed traffic, considering traffic in equilibrium, taking into account vehicle’s micro/mesoscopic characteristics and different lane policies to accommodate AVs. This work provides meaningful insights into AVs distribution across lanes regarding demand and optimal solutions to accommodate AVs.

However, they assumed that all vehicles travel at a constant free-flow speed of \( u \) until reaching their respective critical spacing. Even though their formulation is appropriate, in terms of simplicity, it does not permit adjustments to consider varying driving behaviors among HDVs, and differences in vehicle characteristics (e.g., acceleration/deceleration). To fill this gap, this paper extends the formulation provided in the section 2 of Chen et al. (2017) to account for variations in the spacing characteristics, fostered by the HDVs’ dynamics in the mixed environment. In addition, we derive limit intervals of HDVs acceleration variations in each time step and then, we implement the model to obtain numerical results of the throughput flow and compare them with capacity values from Chen et al. (2017). The proposed analytical formulation provides useful insights into the impacts of the traffic dynamics in the throughput flow, enabling new perspectives on the future complex dynamic of mixed traffic.

2 SINGLE LANE HIGHWAY CAPACITY FORMULATION IN CHEN ET AL. (2017)

Chen et al. (2017), in its section 2, formulate the physical lane capacity of a single lane mixed traffic. Their formulation considers micro/mesoscopic characteristics, such as inter-vehicle spacing of AVs and HDVs, the proportion of AVs per cycle \( (n + m) \), and AV platoon size. They have assumed that both AVs and HDVs travel at a constant free-flow speed of \( u \) until they reach their respective critical spacing. Besides that, they differentiated four critical spacing levels: (1) \( s_0 \) for a HDV (vehicle R) following another HDV, (2) \( \beta^A s_0 \) for the lead vehicle in an AV (vehicle A) platoon, (3) \( \gamma s_0 \) for other AVs in the platoon and (4) \( \beta^R s_0 \) for the first HDV following an AV platoon. Figure 1 illustrates these spacing characteristics.
Another major consideration is that all AVs are platooned and that AV platoons exist periodically, so the traffic stream is periodic with each cycle consisting of one $n$-AV platoon and $m$ HDVs. Under such considerations, the mean critical spacing per cycle ($\bar{S}$) can be expressed as follows.

$$
\bar{S} = \beta^A s_o + (n - 1) \gamma s_o + \bar{m} \left( \beta^R s_o + (m - 1) s_o \right)
$$

Where:

$$
\bar{m} = \begin{cases} 
1, & \text{if } m \geq 1 \\
0, & \text{if } m = 0 
\end{cases}
$$

Equation 1 is re-written as:

$$
\bar{S} = s_o \left( 1 - \alpha \varepsilon \right)
$$

Where:

$$
\alpha = \frac{n}{n + m}
$$

Here, $\alpha$ represents the AV proportion in the traffic stream, and $\varepsilon$ denotes the average gain of critical spacing per AV and is given by the following equation.

$$
\varepsilon = \begin{cases} 
1 - \gamma - \left( \frac{\beta^A - \gamma}{n} + \frac{\beta^R - 1}{n} \right), & \text{if } 0 \leq \alpha < 1 \\
1 - \gamma - \frac{\beta^A - \gamma}{n}, & \text{if } \alpha = 1 
\end{cases}
$$

Then, the capacity ($C$) is derived in Equation 6.

$$
C = \frac{1}{\bar{S}/u} = \frac{u}{s_o(1 - \alpha \varepsilon)} = \frac{C_0}{1 - \alpha \varepsilon}
$$
Where $C_0$ corresponds to the lane capacity with only HDVs. Note that, even though the capacity is a function of $\alpha$ and $\varepsilon$, it is constant in the time domain.

### 3 PROPOSED THROUGHPUT FLOW FORMULATION

In the previous section the spacing characteristics are considered as fixed/deterministic parameters, therefore they are constant over time. In this section, this paper presents a reformulation considering these parameters as functions of time. Firstly, consider $\beta^A(t_{i+1})s_0$, $\beta^R(t_{i+1})s_0$ and $S_j(t_{i+1})$ replacing $\beta^A s_0$, $\beta^R s_0$ and $s_0$, respectively. In this formulation, $t_{i+1}$ is the variable representing time, where $i = 0, 1, 2, 3, \ldots$ and the index “$j$”, in $S_j(t_{i+1})$, accounts for the varying HDVs spacings in the $m$ HDVs queue, thereupon $j = 1, 2, 3, \ldots, m$. Note that “$j$”, when applied to $S_j(t_{i+1})$, will range from 1 to $(m - 1)$, since between $m$ HDVs there are only $(m - 1)$ spacings. Figure 2 shows these parameters.

![Figure 2: Illustration of the proposed inter-vehicle spacing characteristics.](image)

Similar to the previous section, the assumptions that all AVs are platooned and that AV platoons exists periodically, consisting of one $n$-AV platoon and $m$ HDVs, are kept. We consider that all AVs travel at a constant free-flow speed of $u$, which means that $\gamma s_0$ remains constant, but each HDV will have its own respective speed varying with time. The consideration of the $n$-AV platoon remaining at a constant free-flow speed is quite consistent due to the precise future level of automation and because AVs are superior in terms of improving string stability (Talebpour and Mahmassani, 2016). In addition, the start point of the analysis is $t_0$, when all the assumptions outlined in section 2 are satisfied, then, in the subsequent time, $t_1$, the spacing fluctuations start. Notice that we consider that the $n$-AV platoons will operate under steady-state flow while the $m$ HDVs queues are traveling at car-following settings. This does not mean that we are mixing steady-state model with car-following considerations, but rather that we are considering small-scale fluctuations on the dynamics of HDVs’ interactions, which are expected in real traffic.
Firstly, let $S_{Rj}(t_{i+1}), V_{Rj}(t_{i+1})$ and $a_{Rj}(t_{i+1})$ represent the position, the speed, and the acceleration of each HDV ($R_j$), respectively. In this formulation, $a_{Rj}(t_{i+1})$ consist of the primary variables, while $V_{Rj}(t_{i+1})$ and $S_{Rj}(t_{i+1})$ are functions of $a_{Rj}(t_{i+1})$. $V_{Rj}(t_{i+1})$ is given by the following equation.

$$V_{Rj}(t_{i+1}) = V_{Rj}(t_i) + a_{Rj}(t_{i+1}) \Delta t \quad (7)$$

Where $\Delta t = (t_{i+1} - t_i)$ and it is considered constant throughout the analysis. All of the equations, involving “$\Delta t$” in this paper, are only valid for values of “$\Delta t$” sufficiently small. $V_{Rj}(t_{i+1})$ can be re-written as follows:

$$V_{Rj}(t_{i+1}) = u + \Delta t \sum_{k=1}^{i+1} a_{Rj} (t_k) \quad (8)$$

Analogously, $S_{Rj}(t_{i+1})$ is given by Equation 9.

$$S_{Rj}(t_{i+1}) = S_{Rj}(t_i) + [V_{Rj}(t_{i+1})] \Delta t \quad (9)$$

$S_{Rj}(t_{i+1})$ can be re-written as follows:

$$S_{Rj}(t_{i+1}) = S_{Rj}(t_0) + \Delta t \sum_{k=1}^{i+1} V_{Rj} (t_k) \quad (10)$$

Now it is important to extend the last term of Equation 10. Applying Equation 8, and after some algebraic operations, it can be proven that:

$$\sum_{k=1}^{i+1} V_{Rj} (t_k) = (i + 1)u + \Delta t \sum_{k=1}^{i+1} (i + 2 - k)a_{Rj} (t_k) \quad (11)$$

It may seem, for now, that Equation 11 is an unnecessary mathematical exaggeration, but it will prove to be very appropriate for the total formulation in this paper.

3.1 SPACING PARAMETERS FORMULATION

Provided that, $S_j(t_{i+1})$ is easily formulated as follows:

$$S_j(t_{i+1}) = S_{Rj}(t_{i+1}) - S_{R(j+1)}(t_{i+1}) \quad (12)$$
Applying Equation 10 and Equation 11 into Equation 12, and making appropriate arrangements, $S_j(t_{i+1})$ is finally expressed:

$$S_j(t_{i+1}) = s_0 + \Delta t^2 \sum_{k=1}^{i+1} (i + 2 - k)[a_{Rj}(t_k) - a_{R(j+1)}(t_k)]$$ (13)

Equation 13 gives all spacings between each pair of HDVs, in its queue, as a function of time. This equation is appropriate since it is, intentionally, reduced to the accelerations – the primary variables.

Similarly, $\beta^R(t_{i+1})s_0$ is derived:

$$\beta^R(t_{i+1})s_0 = \beta^R(t_i)s_0 + [u - V_{R1}(t_{i+1})] \Delta t$$ (14)

$\beta^R(t_{i+1})s_0$ can be re-written as follows:

$$\beta^R(t_{i+1})s_0 = \beta^R s_0 + [(i + 1)u - \sum_{k=1}^{i+1} V_{R1}(t_k)] \Delta t$$ (15)

Now, employing Equation 11 into Equation 15, $\beta^R(t_{i+1})s_0$ is settled.

$$\beta^R(t_{i+1})s_0 = \beta^R s_0 - \Delta t^2 \sum_{k=1}^{i+1} (i + 2 - k)a_{R1}(t_k)$$ (16)

Equation 16 gives the spacing for the first HDV following an AV platoon as a function of time. Again, this equation is appropriate since it is, intentionally, reduced to $a_{R1}(t_{i+1})$ – a primary variable.

Finally, $\beta^A(t_{i+1})s_0$ is also formulated:

$$\beta^A(t_{i+1})s_0 = \beta^A(t_i)s_0 + [V_{Rm}(t_{i+1}) - u] \Delta t$$ (17)

$\beta^A(t_{i+1})s_0$ can be re-written as follows:

$$\beta^A(t_{i+1})s_0 = \beta^A s_0 + \sum_{k=1}^{i+1} V_{Rm}(t_k) - (i + 1)u \Delta t$$ (18)

Applying Equation 11 into Equation 18, $\beta^A(t_{i+1})s_0$ is given as follows:
\[
\beta^A(t_{i+1})s_0 = \beta^A s_0 + \Delta t^2 \sum_{k=1}^{i+1} (i + 2 - k) a_{Rm}(t_k) \tag{19}
\]

Equation 19 gives the spacing for the lead vehicle in an AV platoon as a function of time. Again, this equation is appropriate since it is, intentionally, reduced to \(a_{Rm}(t_{i+1})\) – a primary variable.

Provided the spacing parameters formulation, this paper proceeds to derive a formulation for the time-varying throughput flow, under the assumptions outlined in this section.

### 3.2 TIME-VARYING THROUGHPUT FLOW FORMULATION

Firstly, similarly to Equation 1, the mean critical spacing per cycle (\(\bar{S}\)) can be expressed as follows.

\[
\bar{S} = \frac{\beta^A(t_{i+1})s_0 + (n - 1) \gamma s_o + \beta^R(t_{i+1}) s_0}{n + m} + \frac{1}{n + m} \sum_{j=1}^{m-1} S_j(t_{i+1}) \tag{20}
\]

Before advancing further, note that \(\bar{m}\) was disregarded. This makes all sense, since this paper addresses a formulation of a mixed traffic environment, therefore \(m \geq 1\). Considering a scenario of \(m = 0\) is not reasonable, because it would not lead to mixed traffic. Then, the last summation, appearing in Equation 20, can be expanded.

\[
\sum_{j=1}^{m-1} S_j(t_{i+1}) = \sum_{j=1}^{m-1} \left[ s_0 + \Delta t^2 \sum_{k=1}^{i+1} (i + 2 - k) [a_{Rj}(t_k) - a_{R(j+1)}(t_k)] \right] \tag{21}
\]

After some algebra, Equation 21 is reduced to the following equation.

\[
\sum_{j=1}^{m-1} S_j(t_{i+1}) = (m - 1)s_0 + \Delta t^2 \sum_{k=1}^{i+1} (i + 2 - k) [a_{R1}(t_k) - a_{Rm}(t_k)] \tag{22}
\]

Employing Equation 16, Equation 19, and Equation 22 into Equation 20, \(\bar{S}\) will be, exactly, equal to Equation 1 (without \(\bar{m}\)). More conveniently it can also assume the form of Equation 3, with \(\alpha\) and \(\epsilon\) being as the same as described by Equation 4 and Equation 5, respectively.

Beyond the mathematical aspect of that, a “physical” interpretation is easy to derive. Notice that all spacing fluctuations, internal to the HDVs queue, is “absorbed” internally, in terms of summation. Additionally, variations in \(S_j(t_{i+1})\) are compensated
by variations in $\beta^R(t_{i+1})s_0$, and variations in $S_{(m-1)}(t_{i+1})$ are compensated by variations in $\beta^A(t_{i+1})s_0$.

Finally, the throughput flow function is derived using the concept of mean time speed and the same procedure applied in section 2. Equation 23 explicitly states the start points for deriving the throughput flow function.

$$Q(t_{i+1}) = \frac{1}{n + m} \left[ \sum_{j=1}^{n} u + \sum_{j=1}^{m} V_{Rj}(t_{i+1}) \right] \frac{1}{s_0 \left( 1 - \alpha \varepsilon \right)}$$

(23)

Using Equation 8, and after some algebraic arrangements, the throughput flow is, finally, given by Equation 24.

$$Q(t_{i+1}) = \frac{C_0}{1 - \alpha \varepsilon} + \frac{\Delta \varepsilon}{(n + m)s_0 \left( 1 - \alpha \varepsilon \right)} \sum_{j=1}^{m} \sum_{k=1}^{i+1} a_{Rj}(t_k)$$

(24)

Equation 24 represents the throughput flows (per cycle) as a function of time. Notice that the first term on the right side of this equation refers to the capacity formula derived by Chen et al. (2017) and shown by Equation 6. The last term on the right side of Equation 24 accounts for variations on the throughput flow fostered by the dynamics of HDVs accelerations.

Notably, the capacity given by Equation 6 is constant over time, on the other hand the throughput flow given by Equation 24 is a function of time. It is worth noting that the divergence (in values) between the capacity given by Chen et al. (2017) and the throughput flow values from Equation 24 is expected to be higher for small values of $\alpha$, and decrease as $\alpha$ increases towards one. This consists of a conjecture that is discussed in section 5 of this paper, and in section 5 we also analyze a particular (and important) case of HDVs’ interactions, where expected operational spacing’s ranges are violated and we have sufficiently small $\beta^A(t_{i+1})s_0$ and $\beta^R(t_{i+1})s_0$.

In section 5, this paper brings up a comparison between these two models. This comparison is provided by a numerical analysis. To do so, it is necessary to set limit intervals for variations of $a_{Rj}(t_{i+1})$ and define a numerical procedure for choosing all $a_{Rj}(t_{i+1})$. For this reason, the next section presents this method.
4 DEFINING THE HDVs ACCELERATION LIMIT INTERVALS

In this section, we define limit intervals for $a_{R_j}(t_{i+1})$ variations. First, three limit intervals are set for $a_{R_1}(t_{i+1})$, and then the generalized limit intervals for $a_{R(j>1)}(t_{i+1})$ are proposed, considering two states of driver awareness.

4.1 LIMIT INTERVALS FOR $a_{R_1}(t_{i+1})$

The dynamics is started by $a_{R_1}$ which is constrained by the interrelation between $R_1$ and the last AV in the platoon, $R_1$ and $R_2$, and the physical constraints. In other words, $a_{R_1}(t_{i+1})$ are limited by the minimum values that $\beta^R(t_{i+1}) s_o$ and $S_1(t_{i+1})$ can assume, and by the maximum acceleration/deceleration variation that $R_1$ can achieve in $\Delta t$.

Considering the interrelation between $R_1$ and the last AV in the platoon, the first limit interval for $a_{R_1}(t_{i+1})$ can be proposed. Equation 25 imposes the following limit interval for $\beta^R(t_{i+1}) s_o$ variations:

$$D_{min} < \beta^R(t_{i+1}) s_o < [D_{max}]_1$$

Where $D_{min}$ is the safe critical spacing and considered equal to $\gamma s_o$ in this work (to be conservative). It is considered as the minimum spacing value hereafter. For this scenario, we consider that $[D_{max}]_1$ is achieved when all $S_j(t_{i+1})$ and $\beta^A(t_{i+1}) s_o$ are equal to the safe critical spacing ($\gamma s_o$). Under this circumstance:

$$[D_{max}]_1 = (\beta^A + \beta^R + m(1 - \gamma) - 1) s_o$$

Considering Equation 16 and the safe critical spacing equal to $\gamma s_o$, Equation 25 becomes:

$$\gamma s_o < \beta^R s_o - \Delta t^2 \sum_{k=1}^{i+1} (i + 2 - k) a_{R_1}(t_k) < (\beta^A + \beta^R + m(1 - \gamma) - 1) s_o$$

Now, this inequality can be re-written as follows:

$$[a_{R_1}(t_{i+1})_{min}]_1 < a_{R_1}(t_{i+1}) < [a_{R_1}(t_{i+1})_{max}]_1$$

Where:

$$[a_{R_1}(t_{i+1})_{min}]_1 = -\frac{s_o}{\Delta t^2} (\beta^A + m(1 - \gamma) - 1) - \sum_{k=1}^{i} (i + 2 - k) a_{R_1}(t_k)$$
\[ [a_{R1}(t_{i+1})_{\text{max}}]_1 = -\frac{s_o}{\Delta t^2} (\gamma - \beta R) - \sum_{k=1}^{i} (i + 2 - k) a_{R1}(t_k) \]  

(30)

Equation 28 is the first limit interval for \( a_{R1}(t_{i+1}) \) variations, considering the interrelation between \( R_1 \) and the last AV in the platoon.

The second limit interval for \( a_{R1}(t_{i+1}) \) is proposed based on the interrelation between \( R_1 \) and \( R_2 \). Equation 31 imposes the following limit interval for \( S_1(t_{i+1}) \) variations:

\[ \gamma s_o < S_1(t_{i+1}) < [D_{\text{max}}]_2 \]  

(31)

For this scenario, we consider that \([D_{\text{max}}]_2\) is reached when \( R_1 \) is closest to the last AV in the platoon, or in other words, when \( \beta R(t_{i+1}) s_o \) is equal to \( \gamma s_o \). Notice that it is reasonable only because we are interested in determining \( a_{R1}(t_{i+1}) \) for now. Under this consideration, it can be proven that \([D_{\text{max}}]_2\) is given as follows:

\[ [D_{\text{max}}]_2 = \beta A s_o - \beta A(t_{i+1}) s_o + \beta R s_o - \gamma s_o + (m - 1) s_o - \sum_{j=2}^{m-1} S_j(t_{i+1}) \]  

(32)

Employing Equation 19, Equation 13 and Equation 22 into Equation 32, and after some arrangements, \([D_{\text{max}}]_2\) is re-written:

\[ [D_{\text{max}}]_2 = (\beta R + 1 - \gamma) s_o - \Delta t^2 \sum_{k=1}^{i+1} (i + 2 - k) a_{R2}(t_k) \]  

(33)

Substituting Equation 33 and Equation 13 into Equation 31 and performing some algebra, the second limit interval for \( a_{R1}(t_{i+1}) \) can be easily derived:

\[ [a_{R1}(t_{i+1})_{\text{min}}]_2 < a_{R1}(t_{i+1}) < [a_{R1}(t_{i+1})_{\text{max}}]_2 \]  

(34)

Where:

\[ [a_{R1}(t_{i+1})_{\text{min}}]_2 = (\gamma - 1) \frac{s_o}{\Delta t^2} - \sum_{k=1}^{i} (i + 2 - k) a_{R1}(t_k) + \sum_{k=1}^{i+1} (i + 2 - k) a_{R2}(t_k) \]  

(35)

\[ [a_{R1}(t_{i+1})_{\text{max}}]_2 = [a_{R1}(t_{i+1})_{\text{max}}]_1 \]  

(36)

Equation 34 is the second limit interval for \( a_{R1}(t_{i+1}) \) variations, observing the interrelation between \( R_1 \) and \( R_2 \).
Finally, the third limit interval for \( a_{R1}(t_{i+1}) \) refers to a physical constraint to which all HDVs are subjected. Equation 37 suggests the following physical limit interval for \( a_{R1}(t_{i+1}) \) variations:

\[
a_{Rj_{\min}} < a_{R1}(t_{i+1}) < a_{R1}(t_i) + \frac{d}{dt}(a_{Rj_{\max}}) \Delta t < a_{Rj_{\max}}
\]  

(37)

Where \( a_{Rj_{\min}} \) represents the maximum possible deceleration and \( a_{Rj_{\max}} \) refers to the maximum possible acceleration. It is worth mentioning that the limit interval given by Equation 37 is applied to all HDVs, since it is a physical constraint.

### 4.2 GENERALIZED LIMIT INTERVALS FOR \( a_R(j>1)(t_{i+1}) \)

To derive the generalized limit interval for \( a_R(j>1)(t_{i+1}) \), two states of driver awareness were designed: (1) state of distraction, and (2) state of alertness. These states of driver awareness are applied to \( R_j \), except for \( R_1 \), because \( R_1 \) does not experience any disturbance on its front, so it leads the HDVs dynamics arbitrarily.

Each state of driver awareness has its own respective reaction time (\( \tau \)), hence \( \tau_D \) and \( \tau_A \) represent the reaction time for the state of distraction and alertness, respectively. Conveniently, we consider that the reaction time is a scalar multiple of \( \Delta t \), therefore \( \tau_D = l_D \Delta t \) and \( \tau_A = l_A \Delta t \). Another consideration is that once an HDV changes from the state of distraction to the state of alertness it remains in this state indefinitely.

To clarify how the process works consider that in \( t_1 \), \( R_1 \) reacts with \( a_{R1}(t_1) \) while \( R_2 \) is in distraction (\( a_{R2}(t_1) = 0 \)). After \( \tau_D \), \( R_2 \) reacts with \( a_{R2}(t_1+l_D) \), but \( R_3 \) is still in distraction; and this process is repeated all the way through the HDVs queue until all HDVs enter the state of alertness. When any HDV enter the state of alertness, it will react every \( \tau_A \) interval of time.

Provide that, and considering the safe critical spacing, the basis “equation” for the transition from the state of distraction to the state of alertness is the following:

\[
\gamma s_o < S_j(t_{1+jl_D}) < S_j(t_{l_D}) + \frac{1}{2} a_{Rj}(t_{l_D}) \Delta t^2 - \gamma s_o
\]  

(38)

Employing Equation 13 into Equation 38, and after some algebraic arrangements, the \( a_R(j>1)(t_{i+1}) \) limit interval for the transition from the state of distraction to the state of alertness is finally derived:

\[
a_{R(j>1)}(t_{1+jl_D})_{\min} < a_{R(j>1)}(t_{1+jl_D}) < a_{R(j>1)}(t_{1+jl_D})_{\max}
\]  

(39)
Where:

\[
a_{R(j>1)}(t_{i+1+j_D})_{\text{min}} = \frac{YS_o}{\Delta t^2} + \sum_{k=1}^{j_D} [a_{Rj}(t_k) - a_{R(j+1)}(t_k)] + a_{Rj}(t_{i+j_D}) - \frac{1}{2} a_{Rj}(t_{j_D}) \\
a_{R(j>1)}(t_{i+1+j_D})_{\text{max}} = \frac{(1-\gamma)S_o}{\Delta t^2} + \sum_{k=1}^{j_D} (j_D + 2 - k)[a_{Rj}(t_k) - a_{R(j+1)}(t_k)] + a_{Rj}(t_{i+j_D})
\] (40) (41)

Regarding the state of alertness, \(R_j\) react every \(\tau_A\) interval of time. The basis “equation” for deriving the limit interval for \(a_{R(j>1)}(t_{i+1})\), reacting under this state, is the following:

\[
\gamma s_o < s_j(t_{i+l_A-1}) < s_j(t_{i+l_A-2}) + \frac{1}{2} a_{Rj}(t_{i+l_A-2}) \Delta t^2 - \gamma s_o
\] (42)

Employing Equation 13 into Equation 42, and after some algebraic arrangements, the \(a_{R(j>1)}(t_{i+1})\) limit interval for this state can be achieved:

\[
a_{R(j>1)}(t_{i+l_A-1})_{\text{min}} < a_{R(j>1)}(t_{i+l_A-1}) < a_{R(j>1)}(t_{i+l_A-1})_{\text{max}}
\] (43)

Where:

\[
a_{R(j>1)}(t_{i+l_A-1})_{\text{min}} = \frac{YS_o}{\Delta t^2} + \sum_{k=1}^{i+l_A-2} [a_{Rj}(t_k) - a_{R(j+1)}(t_k)] + a_{Rj}(t_{i+l_A-1}) - \frac{1}{2} a_{Rj}(t_{i+l_A-2})
\] (44)

\[
a_{R(j>1)}(t_{i+l_A-1})_{\text{max}} = \frac{(1-\gamma)S_o}{\Delta t^2} + \sum_{k=1}^{i+l_A-2} (i + l_A - k)[a_{Rj}(t_k) - a_{R(j+1)}(t_k)] + a_{Rj}(t_{i+l_A-1})
\] (45)

It is important pointing out that for every \(i + 1 \neq i + l_A - 2\), or every \(i + 1 \neq 1 + j_D\), \(a_{R(j+1)}(t_{i+1}) = a_{R(j+1)}(t_i)\) for both states of driver awareness.

Finally, as mentioned in the previous sub-section, all HDVs are subjected to the physical constraint, therefore a general form of Equation 37 should also appear, and this limit interval needs to be checked for every \(a_{R(j>1)}(t_{i+1})\).

\[
a_{Rj_{\text{min}}} < a_{Rj}(t_{i+1}) < a_{Rj}(t_i) + \frac{d}{dt} (a_{Rj(t_{i+1})_{\text{max}}}) \Delta t < a_{Rj_{\text{max}}}
\] (46)

Provided with the limit intervals for \(a_{Rj(t_{i+1})}\) variations, a numerical analysis is possible to be implemented. This numerical analysis is, actually, a simplified mixed traffic flow simulation. The next section shows a comparison of the capacity values
obtained from Equation 6 (Chen et al. (2017)) and the throughput flow values from Equation 24 (proposed formulation).

5 NUMERICAL RESULTS AND CONSIDERATIONS

This section performs a numerical analysis for estimating the capacity and throughput flow with models from section 2 and 3, respectively, considering section 4 to evaluate $a_{R_j(t_{i+1})}$. Three scenarios for the proportion of AVs in the traffic stream are considered: $\alpha = 10\%$, $\alpha = 50\%$, and $\alpha = 70\%$.

Firstly, it is necessary to define the values of all parameters used in this analysis. The safe critical spacing ($\gamma s_o$) is set equal to 7 m (Bang and Ahn, 2019). The time step is chosen appropriately: $\Delta t = 0.6 s$. We consider the maximum possible deceleration ($a_{R_j\text{min}}$) equal to $3.8 \text{ m/s}^2$ and the maximum possible acceleration ($a_{R_j\text{max}}$) equal to $3.3 \text{ m/s}^2$, in accordance with Punzo and Simonelli (2005). In addition, consider $l_D = 3$ and $l_A = 2$, which is in conformity to (Ahn et al., 2004). Lastly, the desired free-flow speed ($u$) is assumed to be $30 \text{ m/s}$ (Bang and Ahn, 2019).

Given that, the first scenario for the proportion of AVs in the traffic stream is implemented, considering $\alpha = 10\%$. Table 1 shows the remaining parameters considered in – and necessary to – the analysis.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta^A$</td>
<td>0.8</td>
<td>dimensionless</td>
</tr>
<tr>
<td>$\beta^A$</td>
<td>1.1</td>
<td>dimensionless</td>
</tr>
<tr>
<td>$n$</td>
<td>1</td>
<td>vehicles</td>
</tr>
<tr>
<td>$m$</td>
<td>9</td>
<td>vehicles</td>
</tr>
<tr>
<td>$\gamma$</td>
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<td>dimensionless</td>
</tr>
<tr>
<td>$s_o$</td>
<td>10</td>
<td>meters</td>
</tr>
</tbody>
</table>

Provided with the parameters’ values, a numerical analysis is implemented. Figure 3 shows the capacity from Equation 6 and the time-varying throughput flow from Equation 24. This analysis was performed for 49 points in the time domain, and as can be seen, the capacity by Chen et al. (2017) is constant while the proposed throughput flow varies with an approximately cyclical behavior. By the nature of the formulation provided in section 4, the capacity would continue to behave similarly to that if the analysis is extended on the time domain.
The maximum throughput flow \(Q(t_{i+1})_{\text{max}}\) is achieved when time is equal to 18.6 s \(t_{31}\), and its value is, approximately, 3.22 vehicles per second. \(Q(t_{i+1})_{\text{max}}\) is about 6.4% greater than \(C\). Similarly, the minimum throughput flow \(Q(t_{i+1})_{\text{min}}\) is reached when time is equal to 24.6 s \(t_{41}\), and its value is, approximately, 2.91 vehicles per second. \(Q(t_{i+1})_{\text{min}}\) is about 3.9% smaller than \(C\).

While Chen et al. (2017)'s model estimates capacity, since they considered that both AVs and HDVs travel at a constant free-flow speed of \(u\) until they reach their respective critical spacing and that this critical spacing corresponds to the capacity, our model considers car-following settings in the \(m\) HDVs queues and their impact on the time-varying throughput flow, taking into account varying driving behaviors among HDVs, and differences in vehicle characteristics (e.g., acceleration/deceleration).

Moreover, the second scenario for the proportion of AVs in the traffic stream is implemented, considering \(\alpha = 50\%\). For this case, we consider the parameters in Table 1, but with the following modification: \(m = n = 5\). With that, Figure 4 shows the capacity from Equation 6 and the time-varying throughput flow from Equation 24.
The maximum throughput flow \( Q(t_{i+1})_{\text{max}} \) is achieved when time is equal to 13.2 s \( (t_{22}) \), and its value is, approximately, 3.54 vehicles per second. \( Q(t_{i+1})_{\text{max}} \) is about 2.6% greater than \( C \). Similarly, the minimum throughput flow \( Q(t_{i+1})_{\text{min}} \) is reached when time is equal to 21.0 s \( (t_{35}) \), and its value is, approximately, 3.35 vehicles per second. \( Q(t_{i+1})_{\text{min}} \) is about 2.8% smaller than \( C \).

Lastly, the third scenario for the proportion of AVs in the traffic stream is implemented, considering \( \alpha = 70\% \). For this case, we consider the parameters in Table 1, but with the following modification: \( m = 3 \) and \( n = 7 \). With that, Figure 5 shows the capacity from Equation 6 and the time-varying throughput flow from Equation 24.

The maximum throughput flow \( Q(t_{i+1})_{\text{max}} \) is achieved when time is equal to 19.8 s \( (t_{33}) \), and its value is, approximately, 3.77 vehicles per second. \( Q(t_{i+1})_{\text{max}} \) is about 1.8% greater than \( C \). Similarly, the minimum throughput flow \( Q(t_{i+1})_{\text{min}} \) is reached
when time is equal to 25.2 s \( t_{42} \), and its value is, approximately, 3.63 vehicles per second. \( Q(t_{i+1})_{\text{min}} \) is about 1.9% smaller than \( C \).

From the figures and values above, it becomes clear that the difference between the maximum and the minimum throughput flow values and the capacity by Chen et al. (2017) is higher for small values of \( \alpha \), and this difference decreases as \( \alpha \) increases. In the transition period, from regular traffic to mixed traffic, values of \( \alpha \) will initially be small, which means that the stream will have less AVs than HDVs, and therefore traffic disturbances will still govern, and flow stability will not be significantly improved. In contrast, as \( \alpha \) evolves, traffic disturbances are reduced, which can improve roadway capacity and flow stability.

Furthermore, the proposed formulations offer new perspectives to the modeling and understanding of mixed traffic since they permit adjustments to consider varying driving behaviors among HDVs and differences in vehicle characteristics. The proposed formulations are promising in this sense because they are all reduced to the acceleration variables, and, consequently, different vehicle characteristics can be employed to the formulations to evaluate their impact to the throughput flow and stability. In addition, considering the formulations in section 4.2, it is possible to consider varying driving behaviors among HDVs only by applying different values for \( \tau_D \) and \( \tau_A \) for each HDV \( (\tau_{Dj} \) and \( \tau_{Aj} \) combined with different levels of driving aggressiveness. In the near future, real-time identification of driving behavior characteristics will be widespread and, considering connected environments, the data generated in the traffic dynamics will serve to establish HDVs control methods.

5.1 PARTICULAR CASE \( (\beta^A(t_{i+1})s_0 < \gamma \) AND \( \beta^R(t_{i+1})s_0 < 1) \)

From Chen et al. (2017), we know that the capacity increases with platoon size if expected operational spacing ranges are considered \( (\beta^A > \gamma, \beta^R > 1, \text{and} \gamma < 1) \). Operational features of AVs are revealed in the literature and supposed in traffic simulations; however, these ranges can be violated in real traffic scenarios, so we can have significantly small \( \beta^A(t_{i+1})s_0 \) and \( \beta^R(t_{i+1})s_0 \), corresponding to \( \beta^A < \gamma, \beta^R < 1 \). While this consists of a particular case in Chen et al. (2017)’s model that should be considered separately in a specific analysis, our formulation captures this possible particular case directly in the model since \( \beta^A(t_{i+1})s_0 \) and \( \beta^R(t_{i+1})s_0 \) are functions of time.
For sufficiently small values of $\beta^A$ and $\beta^R$ ($\beta^A < \gamma$, $\beta^R < 1$) the lane capacity will decrease with platoon size, which makes this a particularly important case. This can be achieved when we consider that $R_1$ exhibits reckless driving behavior and $R_m$ performs over-cautious driving behavior. This pattern can potentially degrade the capacity. From a pure-math perspective, considering reckless $R_1$ and over-cautious $R_m$ as independent events, the probability of the occurrence of this specific pattern in a cycle is very small, nonetheless in real traffic this probability increases considerably.

First, for a given value of $\alpha$ in the traffic stream, in real traffic each cycle will not necessarily have the same $\alpha$, which for Chen et al. (2017)'s model does not make considerable differences, because even if considering varying $\alpha$ among cycles, from a multi-cycle perspective the probability of the occurrence of this specific case will be still negligible. On the other hand, in real traffic, for cycles with greater $\alpha$, which means smaller $m$-HDVs queues, the probability of having a reckless $R_1$ and an over-cautious $R_m$ in these cycles increases substantially, and this can lead to a decrease in the capacity.

Second, in real traffic, we expect that reckless drivers will exhibit aggressive passing maneuvers, with risky lane-changing maneuvers and disregarding safe critical spacing, while over-cautious drivers will tend to accelerate less aggressively and maintain safer spacings. Therefore, even with longer $m$-HDVs queues, in practice, the probability to have a reckless $R_1$ and an over-cautious $R_m$, in an $m$-HDVs queue, is greater than expected from a pure-math perspective.

Obviously, this discussion transcends the scope of the assumptions and resulting formulations presented in this work, since we are not considering lane-changing or passing maneuvers in our model, however, this discussion sheds light on an important case that can potentially degrade the capacity. In this sense, our formulations can incorporate this event by employing specific reaction times and acceleration patterns to simulate a scenario with reckless $R_1$ and over-cautious $R_m$, with little adjustments in section 4.

6 CONCLUSION AND DISCUSSION

This study derives formulations for estimating time-varying throughput flows in single-lane highways based on a car-following model with automated and human-driven vehicles. Initially, the spacing characteristics formulas are derived, in terms of acceleration, and then the throughput flow is formulated to consider the HDVs dynamics. This paper further develops a formulation for the limit intervals of variation for HDVs’
accelerations, delineating two states of driver awareness: state of distraction and state of alertness.

Accordingly, the formulations proposed in this paper are convenient in a physical and mathematical point of view, not only because they are discrete formulations, which is favorable for computational implementations, but also for the reason that they are easy to interpret in terms of physical dynamics of mixed traffic. Moreover, the proposed formulations showed to be promising to the understanding of mixed traffic since they permit adjustments to consider varying driving behaviors among HDVs and differences in vehicle characteristics. Therefore, this study provides useful insights into the impacts of the traffic dynamics in the throughput flow, enabling new perspectives on the future complex dynamic of mixed traffic.

Some extensions of the present study can be suggested as future investigations. First, this study needs to be balanced with empirical/experimental studies of AV behavior and the interactions between AVs and HDVs in mixed traffic. Such a comparison serves to either validate the formulations or to incite new perspectives and questioning. Second, dynamics of traffic disturbance (e.g., merging, diverging and lane-changing) can be incorporated to the formulations. These traffic disturbance phenomena interfere in the platooning process and in the dynamic of HDVs interrelations, directly affecting the throughput flow and stability. Throughput flow formulation, considering these issues, is fundamental to obtain a basic model to understand some challenging features of mixed traffic, and it is left for future research.

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REFERENCES


