The complex numbers of the matricial view point

Os números complexos do ponto de vista matricial

DOI:10.34117/bjdv7n7-062

Recebimento dos originais: 05/06/2021
Aceitação para publicação: 05/07/2021

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ABSTRACT
The present paper aims to present a study carried out on complex numbers in a different way from the traditional one, which occurs through ordered pairs or in an algebraic form. For this, we use matrix algebra techniques. Here we present some basic properties of the of complex numbers in their matrix form, comparing them with those found in basic mathematical literature.

Keywords: Matrix, Complex number, Polar form, Moivre's Formula.

RESUMO

O presente trabalho visa apresentar um estudo realizado sobre números complexos de uma forma diferente da tradicional, que ocorre através de pares ordenados ou em forma algébrica. Para isso, utilizamos técnicas de álgebra matricial. Aqui apresentamos algumas propriedades básicas dos números complexos em sua forma matricial, comparando-os com aqueles encontrados na literatura matemática básica.

Palavras-Chave: Matriz, Número Complexo, Forma Polar, Fórmula de Moivre.

1 INTRODUCTION

We begin the text by remembering that complex numbers can be defined as being ordered pairs (a, b), where a and b are real numbers and the set of these ordered pairs, denoted by C, is a field with the operations of addition and multiplication defined by $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b).(c, d) = (ac - bd, bc + ad)$, respectively (See [2]). Now, the field R is immersed in C through isomorphism $a \mapsto (a, 0)$ and being by definition $i = (0, 1)$, so $(a, b) = a+bi$. and being by definition using matrix algebra,it was possible to study in [6] the properties of complex numbers from this point of view. We see in [7] that we can begin the study of complex numbers from the most elementary construction that we can have: the solution of equation $x^2 = -1$. We know that in R there is no solution to this equation, then the need arises to define a new set where it has a solution. In [5] the complex numbers are presented in the form $z = a + bi$, where i is called the imaginary number that satisfies the property $i^2 = -1$, where a and b belong to the set of real numbers, which we will denote by R. Natural approach to basic education. The complex numbers set, denoted here by C, where are defined the operations of addition and multiplication as follows: Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ in C, then: $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$, while $z_1 \cdot z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_1)i$.

Now let's consider the following matrix equation $X \cdot X = -I$, where X is a matrix of order 2 with real entries and I is the identity matrix.

The matrix $\mathfrak{I} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is solution of the matrix equation. Indeed:

$$\mathfrak{I}^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I.$$

Our focus in presenting the theory will be algebraic, unlike the approaches adopted in some texts, which are focused on the geometric aspect, see you [3], by example. The paper is organized as follow: In the section 2, we present a specific class of matrices of order two, where the product operation is commutative. We also present an application that identifies the set of complex numbers with this class of matrices, and finally we study the matrix division operation. In section 3, we present the concept of conjugate in this framework, and its properties. A norm for this class of matrices is presented, along with its properties. For example, triangle inequality. Finally, the polar form of a complex number and the Moivre's formula are presented.

2 PROPERTIES OF OPERATIONS IN M2(R)

Let $M_2(R) = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}$ be, it is obvious that $M_2(R) \neq \emptyset$, because the identity matrix $I \in M_2$(R). We have that the operations of sum and product of matrices are closed in M2 (R), because:

Given the matrices $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $B = \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \in M_2(\mathrm{R})$, then:

$$A + B = \begin{bmatrix} a+c & -(b+d) \\ b+d & a+c \end{bmatrix} \text{ and } A \cdot B = \begin{bmatrix} ca-db & -(cb+da) \\ cb+da & ca-db \end{bmatrix}.$$

Furthermore, we have that $A \cdot B = B \cdot A$, for all $A, B \in M_2(R)$, then $M_2(\mathrm{R})$ has, in relation to the product, the commutative property.

Theorem 1. The function $\Phi : \mathrm{C} \rightarrow M_2(\mathrm{R})$ definide by $\Phi(a + bi) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, satisfies the following properties: Let $z_1, z_2 \in \mathrm{C}$, then:

1. $\Phi(z_1 + z_2) = \Phi(z_1) + \Phi(z_2);$
2. $\Phi(z_1 \cdot z_2) = \Phi(z_1) \cdot \Phi(z_2);$
3. $\Phi(z^{-1}) = \Phi(z)^{-1}$, *para todo* $z \in \mathbb{C}^*;$
4. $\Phi(1) = I;$
5. $\Phi(i) = \mathfrak{J};$
6. *The function* Φ *is a bijective.*

Proof. We'll proof only the statement (3) and (6). Let $z = a + bi \neq 0$ be, calculating the determinant of the matrix $\Phi(z)$, we obtain that $Det\ \Phi(z) = a_2 + b_2 \neq 0$, So there is $\Phi(z)^{-1}$.

$$\Phi(z^{-1}) = \Phi\left(\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i\right) = \begin{bmatrix} \frac{a}{a^2+b^2} & \frac{b}{a^2+b^2} \\ \frac{-b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{bmatrix} = \Phi(z)^{-1}.$$

To prove (6), let us first observe that fact function is surjective, and this follows directly from the function's own definition. Now, we go to prove that the fuction Φ is injective. Let $z_1 = a + bi$ and $z_2 = c + di$, complex numbers, such that $\Phi(z_1) = \Phi(z_2)$, that is,

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}.$$

So, $a = c$ and $b = d$, and then $z_1 = z_2$. Furthermore, we conclude that Φ is bijective.

Therefore, for all complex number $z = a + bi$ is associated an matrix

$$Z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = aI + b\mathfrak{J}.$$

Thus, we can identify C with the set $\{aI + b\mathrm{J}: a, b \in R\}$. Therefore, we have a representation of the complex numbers by matrices. The matrix aI is called the real part of the complex number matrix Z, denoted by $Re(Z)$, while bI is called the imaginary part, denoted by $Im(Z)$. The sum and product operations in $C = \{aI + b\mathrm{J};\ a, b \in R\}$ are defined as follows: Let $Z_1 = aI + b\,\mathrm{J}$, $Z_2 = cI + d\,\mathrm{J} \in C$,

i. Addition: $Z_1 + Z_2 = (a + c)I + (b + d)\,\mathrm{J}$

ii. *Multiplication:* $Z_1 \bullet Z_2 = (ac - bd)I + (ad + bc)\,\mathrm{J}$.

If $Det(Z_2) \neq 0$, then:

$$Z_2^{-1} = \frac{1}{Det(Z_2)} \cdot \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \frac{1}{Det(Z_2)} Z_2^T, \tag{1}$$

where Z_2^T is the transpose matrix of the Z_2. Thus, we present a version for the division of a complex number Z_1 by another Z_2, where $Det\,(Z_2) \neq 0$:

$$\frac{Z_1}{Z_2} := \frac{Z_1 \cdot Z_2^T}{Det(Z_2)}. \tag{2}$$

3 CONJUGATE AND MATRIX POLAR FORM

Let $Z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ be, we define the complex conjugate of Z by $\overline{Z} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

It proves easily, using operations with matrices, the following theorem, whose results coincide with the classical case.

Theorem 2. If Z, Z_1 and Z_2 are complex numbers, so:

1. $Z + \overline{Z} = 2Re(Z)$;
2. $Z - \overline{Z} = 2Im(Z)\mathfrak{J}$;
3. $Z = \overline{Z} \Leftrightarrow Z = Re(Z)$;
4. $\overline{Z_1 + Z_2} = \overline{Z_1} + \overline{Z_2}$;
5. $\overline{Z_1 Z_2} = \overline{Z_1}\ \overline{Z_2}$.

Let the complex $Z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, the norm of Z is defined as being the the not negative real number $||Z|| = \sqrt{a^2+b^2}$ (see [4]).

Note that $Det(Z) = ||Z||^2$ and being $Z^T = Z$, so

$$\frac{Z_1}{Z_2} := \frac{Z_1 \cdot \overline{Z_2}}{||Z_2||^2},$$

which is analogous to the division of usual complexes

Theorem 3. Let Z be a complex, then:.

1. $||Z|| = 0 \Leftrightarrow Z = 0$;
2. $||Z|| = ||\overline{Z}||$;
3. $||Re(Z)|| \leq ||Z||$;
4. $||Im(Z)|| \leq ||Z||$;
5. $||Z||^2 = ||Z \cdot \overline{Z}||$.
6. $||Z_1 \cdot Z_2|| = ||Z_1|| \cdot ||Z_2||$;

Proof. Let $Z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ a complex, we'll proof only the statements (3.) and (5.):

3. The $Re(Z) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ so $||Re(Z)|| = \sqrt{a^2} \leq \sqrt{a^2+b^2} = ||Z||$.
5. Being, $Z \cdot \overline{Z} = \begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix}$, then $||Z \cdot \overline{Z}|| = a^2+b^2 = ||Z||^2$. □

Lemma 4. Let $Z_1, Z_2 \in M_2(R)$ be, then:

$$|ac+bd| \leq ||Z_1|| \cdot ||Z_2||.$$

Proof. Let $Z_1 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and $Z_2 = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$.

Adding $a^2c^2 + b^2d^2$ on both sides of the inequality $2adbc \leq a^2d^2 + b^2c^2$, we get:

$$\begin{array}{rcl} a^2c^2 + 2adbc + b^2d^2 & \leq & a^2d^2 + a^2c^2 + b^2c^2 + b^2d^2 \\ (ac+bd)^2 & \leq & (a^2+b^2)(c^2+d^2) \\ |ac+bd| & \leq & ||Z_1|| \cdot ||Z_2||. \end{array}$$

Theorem 5. Let Z_1 and Z_2 complex numbers, then:

1. $||Z_1 + Z_2|| \leq ||Z_1|| + ||Z_2||$; *(Triangle inequality.)*

2. $||Z_1 - Z_2|| \geq |\,||Z_1|| - ||Z_2||\,|$.

Proof. Let $Z_1 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ e $Z_2 = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$.

1. Being, $Z_1 + Z_2 = \begin{bmatrix} a+c & -(b+d) \\ b+d & a+c \end{bmatrix}$, then

$$\begin{aligned} ||Z_1 + Z_2||^2 &= ||Z_1||^2 + ||Z_2||^2 + 2(ac+bd) \\ &\leq ||Z_1||^2 + ||Z_2||^2 + 2|ac+bd| \end{aligned}$$

By lemma 4, we have:

$$||Z_1 + Z_2||^2 \leq ||Z_1||^2 + 2||Z_1|| \cdot ||Z_2|| + ||Z_2||^2 = (||Z_1|| + ||Z_2||)^2.$$

Hence, $||Z_1 + Z_2|| \leq ||Z_1|| + ||Z_2||$.

2. Since $Z_1 - Z_2 = \begin{bmatrix} a-c & -(b-d) \\ b-d & a-c \end{bmatrix}$, then

$$\begin{aligned} ||Z_1 - Z_2||^2 &= ||Z_1||^2 + ||Z_2||^2 - 2(ac+bd) \\ &\geq ||Z_1||^2 + ||Z_2||^2 - 2|ac+bd| \end{aligned}$$

Using the lemma 4, we thus conclude that

$$||Z_1 - Z_2||^2 \geq ||Z_1||^2 - 2||Z_1|| \cdot ||Z_2|| + ||Z_2||^2 = (||Z_1|| - ||Z_2||)^2.$$

Thus, $||Z_1 - Z_2|| \geq |\,||Z_1|| - ||Z_2||\,|$. □

Remark 1. *The following properties are valid,* $||Z_1 - Z_2|| \leq ||Z_1 + Z_2||$ *and* $||Z_1 + Z_2|| \geq |\,||Z_1|| - ||Z_2||\,|$, *and is easy to verify using the theorem 5.*

We will now introduce the matrix version of the **polar form of a complex number**. Let $r = \sqrt{x^2 + y^2}$ be, $x = r\cos\theta$ e $y = r\sin\theta$, so

$$Z = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix} = r\cos\theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + r\sin\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = r\cos\theta\, I + r\sin\theta\, \mathfrak{J}.$$

Note that for $r = 1$, we have the matricial complex number $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, wich is the **rotation matrix**.

Now, let's get the Moivre's formula in its complex matricial form: Let $Z = r\cos\theta I + r\sin\theta\, J$ be a complex matrix number in its polar form, then there is a complex number $z = r\cos\theta + r\sin\theta\, i$ such that $Z = \Phi(z)$. Then for all $n \in N$, we have

$$\begin{aligned}
Z^n &= \Phi(z)^n \\
&= \Phi(z^n) \\
&= \Phi(r^n\cos(n\theta) + r^n\sin(n\theta)i) \\
&= \Phi(r^n\cos(n\theta)) + \Phi(r^n\sin(n\theta)i) \\
&= r^n\cos(n\theta)I + r^n\sin(n\theta)\mathfrak{J}; \\
&= r^n[\cos(n\theta)I + \sin(n\theta)\mathfrak{J}]. \qquad (3)
\end{aligned}$$

This is the matricial form of Moivre's formula. Example 1. Let $Z_0 = r_0\cos\theta_0 I + r_0\sin\theta_0 J$ be a matrix complex number. We will solve the matrix equation $Z^n = Z_0$, $n \geq 1$. By Moivre's formula, we have

$$\begin{bmatrix} r^n\cos n\theta & -r^n\sin n\theta \\ r^n\sin n\theta & r^n\cos n\theta \end{bmatrix} = \begin{bmatrix} r_0\cos\theta_0 & -r_0\sin\theta_0 \\ r_0\sin\theta_0 & r_0\cos\theta_0 \end{bmatrix}.$$

So, $r^n\cos n\theta = r_0\cos\theta_0$ *and* $r^n\sin n\theta = r_0\sin\theta_0$. *Therefore,*

$$r^n = r_0, \quad \cos n\theta = \cos\theta_0 \quad and \quad \sin n\theta = \sin\theta_0.$$

So, $\theta_k = \dfrac{\theta_0 + 2k\pi}{n}$, *where* $k \in \mathbb{Z}$, *are the solutions of the system of trigonometric equations. So, the n differents solutions of matrix equation are given by*

$$Z_k = \sqrt[n]{r_0}[\cos(\frac{\theta_0 + 2k\pi}{n})I + \sin(\frac{\theta_0 + 2k\pi}{n})\mathfrak{J}], \; where \; 0 \leq k \leq n-1.$$

Note that for k values greater than n-1 the roots are repeated.

Example 2. *We will calculate the nth roots of unity. Let* $Z^n = I$, *where* $n \geq 1$, *then* $\theta = \dfrac{2k\pi}{n}$, *where* $0 \leq k \leq n-1$. *So, the nth roots of unity are:*

$$\begin{aligned}
Z_0 &= I, \\
Z_1 &= \cos\frac{2\pi}{n}I + \sin\frac{2\pi}{n}\mathfrak{J}, \\
&\vdots \\
Z_{n-1} &= \cos\frac{2\pi}{n}(n-1)I + \sin\frac{2\pi}{n}(n-1)\mathfrak{J}.
\end{aligned}$$

4 CONCLUSION

In this paper, we present the matrix form of numbers complex. In it, it was possible to observe the properties of complex numbers from a different point of view than the one studied in basic education. It is expected that this work will contribute significantly, not only with the theoretical content presented, but also to awaken in students and other interested parties, the interest in studying the complex numbers through matrix representation and seeking applications of the theory.

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